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Critical phenomena in field theories at finite temperature

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Abstract. It is shown how renormalization-group techniques which have been successfully applied to critical phenomena in condensed-matter systems can be adapted to the study of phase transitions in field theories at finite temperature. Explicit calculations are described for the simplest case of a self-interacting scalar field. A systematic method of estimating the transition temperature to all orders of renormalized perturbation theory is given, and the two-loop contribution is found. It appears, however, that there is an additional, non-perturbative contribution which is not determined in the present work. An improved approximation to the effective potential is obtained, in which the renormalization group is used to resum the infrared singularities of perturbation theory. This shows that the transition is of second order, disproving recent claims to the contrary. Gauge theories are discussed qualitatively. While application of the techniques described in this paper will probably show that the transitions in some gauge theories are of first order, it is argued that the order of these transitions can probably not be detetermined reliably using methods currently available.

1. Introduction

It has been believed for some time that the symmetry which is often described as being spontaneously broken in gauge theories such as the standard model would be restored at sufficiently high temperature (Kirzhnits and Linde 1972, Weinberg 1974, Dolan and Jackiw 1974). At a certain critical temperature, the system would undergo a phase transition, analogous to that which is routinely observed in superconductors. Such phase transitions have important cosmological implications, in connection both with the inflationary scenario (Guth 1981, Linde 1982, Albrecht and Steinhardt 1982, Abbott and Pi 1986) and with the possibility of sphaleron-induced baryon-number violation in the standard electroweak theory (Klinkhamer and Manton 1984, Kuzmin *et al* 1985, Arnold and McLerran 1987, McLerran 1989). The study of these transitions in their true dynamical setting is rather difficult (see, for example, Lawrie (1988, 1989, 1992)), but an obvious prerequisite is to understand the behaviour of these theories in thermal equilibrium. In particular, it is important to know whether the phase transitions are of first order (discontinuous) or second order (continuous) and to be able to estimate the critical temperatures at which they occur.

A somewhat simpler exercise is to study the transition in scalar field theory, which is analogous to the Curie transition in a ferromagnet. This is valuable both as a test bed for methods of analysis to be applied to theories of more direct physical interest, and because scalar fields appear in the Higgs sector of many gauge theories. It appears from the recent literature that even this simpler model is not as well (or, perhaps, as widely) understood as it might be. Arnold (1992), for example, asserts that no perturbative method is known for calculating the critical temperature beyond the one-loop result of Dolan and Jackiw

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(1974), on account of the infrared divergences which arise in multi-loop diagrams. It does indeed seem to be true that no systematic method for calculating the critical temperature has been developed, and this is surprising, since the techniques needed to control the infrared behaviour have been known for more than twenty years. Carrington (1992), following earlier work of Takahashi (1985), has obtained approximations to the finite-temperature effective potentials of both the scalar theory and the standard model, and claims that the transitions in both theories are of first order (see also Anderson and Hall (1992), Dine *et al* (1992)). In the case of the scalar theory, it would be most surprising if this were true, since this theory ought to lie in the same universality class as the three-dimensional Ising ferromagnet, which has a second-order transition.

The purpose of the present work is to address both these issues and, more generally, to set out what we believe to be the correct way of investigating critical phenomena in finite-temperature field theory. The key to controlling infrared divergences is the renormalization-group approach to critical phenomena, due largely to Wilson (Wilson and Kogut 1974; see also the review articles in Domb and Green (1976)) which has been elaborated in a field-theoretic context by Brézin *et al* (1976, Zinn-Justin 1989) and others. Indeed, condensed-matter systems which undergo second-order phase transitions can be represented, close to their critical points, by an effective Ginzburg–Landau–Wilson model, which is equivalent to a three-dimensional scalar field theory. The analysis of critical behaviour in this model is reviewed in section 2, where we emphasize a point of view which generalizes readily to thermal field theory. The methods we use for doing perturbative calculations in finite-temperature scalar field theory are described in section 3, while section 4 describes a systematic method for calculating the perturbative contribution to the critical temperature to any order and exhibits the two-loop result. We find, however, that there is an isolated contribution, whose value cannot be ascertained, but which can probably be set to zero within perturbation theory. In section 5, we describe how the renormalization-group analysis which is well known for Ginzburg–Landau–Wilson models can be adapted (in what we believe to be a novel way) for use in thermal field theory, and obtain the one-loop effective potential of the scalar theory. What the one-loop approximation means depends on how one decides to organize the perturbation theory, and our approximation is, essentially, equivalent to that obtained by Carrington. However, we show that when infrared divergences are properly resummed, there is no evidence of a first-order transition.

The renormalization-group analysis uses, in intermediate stages, a renormalized mass and coupling constant which are only indirectly related to those which parametrize the zero-temperature theory in physical terms. To illustrate how finite-temperature properties of the theory can be calculated in terms of the temperature and of zero-temperature parameters, we obtain in section 6 an explicit expression for the expectation value of the scalar field. We find, as expected, that it approaches zero continuously as $T \to T_c$ with the same power-law behaviour that characterizes the magnetization of a ferromagnet.

We give no explicit calculations for gauge theories, but in section 7 we discuss qualitatively what might be expected. In $(3+1)$-dimensional theories where symmetry is spontaneously broken via the Higgs mechanism, we expect that the methods advocated in this paper will, typically, indicate the occurrence of first-order transitions. On the other hand, we argue that such indications are not necessarily reliable, and that no method known to us can settle the issue.

Finally, our conclusions are summarized in section 8.

2. Critical phenomena in Ginzburg–Landau–Wilson models

The archetypical example of a system undergoing a second-order phase transition is an Ising-like ferromagnet. Near its critical point, such a system can be represented as a Euclidean scalar field theory, whose action

$$S = \int \mathrm{d}^d x \left[\frac{1}{2}(\nabla\phi_0)^2 + \frac{1}{2}\mu_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4\right] \tag{2.1}$$

approximates the reduced Hamiltonian βH. The field ϕ_0 represents the spin density of the magnet, while μ_0^2 is linear in temperature and λ_0 is a constant. Subscripts '0' in (2.1) distinguish the bare quantities from their renormalized counterparts introduced below, and d is the spatial dimensionality. In contrast to a genuine quantum field theory, the bare mass is the primary temperature-dependent variable, and the transition between ordered and disordered phases occurs at a value $\mu_{0c}^2(\lambda_0)$ for which the inverse propagator at zero momentum vanishes:

$$\Gamma_0^{(2)}(p^2 = 0; \lambda_0, \mu_{0c}^2) = 0. \tag{2.2}$$

The value of μ_{0c}^2 can be determined formally in perturbation theory by writing, say, $\mu_0^2 = \mu_{0c}^2 + r_0$, using r_0 as the squared mass in the unperturbed propagator and treating μ_{0c}^2 as a counterterm which subtracts the (ultraviolet-divergent) value of each term at $p^2 = r_0 = 0$. Then r_0 is proportional to $(T - T_c)$, where T_c is the exact critical temperature. In the context of dimensional regularization, it is actually consistent and convenient to set $\mu_{0c}^2 = 0$. We shall not do this, however, because in finite-temperature field theory the critical temperature must be found by extracting a finite, temperature-dependent contribution to μ_{0c}^2, and it will be important to keep track systematically of all the potential divergences.

Below four dimensions, individual terms of sufficiently high order in the perturbative expansion of $\Gamma^{(2)}(0; \lambda_0, r_0)$ diverge in the infrared limit $r_0 \to 0$. Of particular interest is the case $d = 3$, where all contributions beyond two-loop order diverge. It is therefore not immediately apparent that $\Gamma_0^{(2)}$ really vanishes at $r_0 = 0$. Nevertheless, it does vanish, and one finds for small r_0

$$\Gamma_0^{(2)} \simeq r_0^{\gamma} \tag{2.3}$$

where γ is a universal exponent, given by $\gamma \approx 1.24$ in three dimensions. To establish this result, and to estimate the value of γ, one uses the renormalization group to control infrared divergences, by relating the propagator of the near-massless theory to that of a massive theory.

We define a renormalized theory by introducing a renormalized field, $\phi(x)$ and renormalized parameters μ^2 and λ, related to the bare quantities by

$$\phi_0(x) = Z_\phi^{1/2}(\lambda)\phi(x) \tag{2.4}$$

$$\mu_0^2 = \mu_{0c}^2 + Z_\mu(\lambda)\mu^2 \tag{2.5}$$

$$\lambda_0 = Z_\lambda(\lambda)\kappa^{4-d}\lambda \tag{2.6}$$

where κ is an arbitrary parameter with the dimension of mass, ensuring that λ is dimensionless. From (2.6), we see that λ is independent of μ^2 and (2.5) shows that μ^2

is directly proportional to r_0, and thus to $T - T_c$. The conditions which might be used to determine the renormalization factors Z_ϕ, Z_μ and Z_λ will be discussed below.

In the usual way, the fact that the unrenormalized Green functions are independent of the renormalization scale κ leads to a renormalization group equation for the renormalized one-particle irreducible functions, namely

$$\left[\kappa\frac{\partial}{\partial\kappa} + W(\lambda)\frac{\partial}{\partial\lambda} + \sigma(\lambda)\mu^2\frac{\partial}{\partial\mu^2} - \frac{N}{2}\eta(\lambda)\right]\Gamma^{(N)}(p; \lambda, \mu^2, \kappa) = 0 \qquad (2.7)$$

where

$$W(\lambda) = \kappa\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\kappa}\right)_{\lambda_0,\mu_0} \qquad (2.8)$$

$$\sigma(\lambda) = \frac{\kappa}{\mu^2}\left(\frac{\mathrm{d}\mu^2}{\mathrm{d}\kappa}\right)_{\lambda_0,\mu_0} \qquad (2.9)$$

$$\eta(\lambda) = \kappa\left(\frac{\mathrm{d}\ln Z_\phi}{\mathrm{d}\kappa}\right)_{\lambda_0,\mu_0}. \qquad (2.10)$$

Solution of this equation by the method of characteristics together with dimensional analysis yields, for $N = 2$ and $p = 0$, the relation

$$\Gamma^{(2)}(0; \lambda, \mu^2, \kappa) = P(\xi)(\xi\kappa)^2\Gamma^{(2)}(0; \bar{\lambda}(\xi), \bar{\mu}^2(\xi)/(\xi\kappa)^2, 1) \qquad (2.11)$$

where $\bar{\lambda}(\xi)$ is the solution of

$$\xi\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}\xi} = W(\bar{\lambda}) \qquad (2.12)$$

with the initial condition $\bar{\lambda}(1) = \lambda$, while $\bar{\mu}^2(\xi)$ and $P(\xi)$ satisfy similar equations, which can be integrated to give

$$\bar{\mu}^2(\xi) = \mu^2\exp\left[\int_1^\xi\frac{\mathrm{d}\xi'}{\xi'}\sigma(\bar{\lambda}(\xi'))\right] \qquad (2.13)$$

$$P(\xi) = \exp\left[-\int_1^\xi\frac{\mathrm{d}\xi'}{\xi'}\eta(\bar{\lambda}(\xi'))\right]. \qquad (2.14)$$

To regularize the infrared divergence at $\mu^2 = 0$, we may choose the free parameter ξ to satisfy the condition

$$\bar{\mu}^2(\xi) = (\xi\kappa)^2 \qquad (2.15)$$

giving

$$\Gamma^{(2)}(0; \lambda, \mu^2, \kappa) = P(\xi)(\xi\kappa)^2\Gamma^{(2)}(0; \bar{\lambda}(\xi), 1, 1). \qquad (2.16)$$

In view of (2.13) and (2.15), we expect ξ to vanish in the infrared limit $\mu^2 \to 0$. The utility of our formal manipulations now rests on the fact that, given an appropriate renormalization prescription, $\bar{\lambda}(\xi)$ approaches a fixed-point value λ^* in this limit. For then $\Gamma^{(2)}(0; \lambda^*, 1, 1)$

is a finite constant, and the infrared singularity is isolated in the prefactors. Assuming the existence of an infrared-stable fixed point, we find

$$\Gamma^{(2)}(0; \lambda, \mu^2, \kappa) \sim \mu^{2\gamma} \tag{2.17}$$

where

$$\gamma = (2-\eta)/(2-\sigma). \tag{2.18}$$

with $\sigma = \sigma(\lambda^*)$ and $\eta = \eta(\lambda^*)$.

Since $\mu^2 \propto r_0$, we can now establish (2.3) and find an estimate for γ, provided that the appropriate fixed point can be located in perturbation theory. To do this, we need a small parameter, ϵ, such that $\lambda^* = O(\epsilon)$. It was discovered by Wilson and Fisher (1972) that the appropriate small parameter is

$$\epsilon = 4 - d. \tag{2.19}$$

In four dimensions, infrared divergences are logarithmic. A double power series expansion of Green functions in powers of λ_0 and ϵ leads to a sum of powers of $\ln r_0$, and the role of the renormalization group is to resum these logarithms into the overall factor r_0^γ, with corrections involving higher powers of r_0. If this procedure is to be carried out systematically, it is clearly necessary to construct a renormalized theory whose Green functions remain finite in four dimensions ($\epsilon = 0$), even though we may eventually be interested in three dimensions ($\epsilon = 1$). Thus, our renormalization prescription must be such as to remove the ultraviolet divergences which appear in four dimensions, which is, of course, the usual field-theoretic situation.

Many renormalization schemes would serve our immediate purpose, but we adopt here a scheme which will be especially useful in the analysis of thermal field theory. To ensure that the renormalization constants Z_i are independent of μ^2, we consider the theory with $\mu^2 = \kappa^2$ and impose on the renormalized Green functions the conditions

$$\Gamma^{(2)}(0; \lambda, \kappa^2, \kappa) = \kappa^2 \tag{2.20}$$

$$\left.\frac{\partial}{\partial p^2}\Gamma^{(2)}(p; \lambda, \kappa^2, \kappa)\right|_{p^2=0} = 1 \tag{2.21}$$

$$\Gamma^{(4)}(0; \lambda, \kappa^2, \kappa) = \kappa^\epsilon \lambda. \tag{2.22}$$

The Z_i defined in this way are, by dimensional analysis, functions only of λ, and will serve also to remove divergences from the theory with other values of μ^2. It would be superfluous to record details of the explicit implementation of this scheme in the present context. Calculations using a variety of renormalization schemes may be found in the literature (see, for example, Brézin *et al* (1976), Lawrie (1976), Zinn-Justin (1989)). One finds that the fixed point does indeed exist. Although the functions $W(\lambda)$, $\sigma(\lambda)$ and $\lambda^*(\epsilon)$ are all scheme-dependent, the expansion of γ in powers of ϵ is scheme-independent and has the form $\gamma = 1 + \epsilon/6 + O(\epsilon^2)$.

A renormalization scheme such as that given by (2.20)–(2.22) can alternatively be implemented by setting $\epsilon = 1$ and calculating directly in three dimensions (Parisi 1980). There are some obvious pitfalls. For example, λ^* is not particularly small, so low orders of perturbation theory do not give very accurate results. Also, while the coefficients in the ϵ

expansion for a quantity such as γ are independent of the chosen renormalization scheme, the truncation errors incurred at a given order of perturbation theory in a fixed-dimension calculation are scheme-dependent. Suppose, moreover, that we wish to check the equality of the two sides of (2.16). Within the ϵ expansion, each side can be expanded as a double power series in λ and ϵ, and the results must agree. With ϵ fixed at 1, however, the left-hand side can be expanded only in powers of λ and the right-hand side in powers of $\bar{\lambda}$. These are two different expansions, which cannot be expected to agree term by term. Despite these difficulties, the two methods agree well when pursued with sufficient vigour, at least in respect of the estimation of critical exponents. The results reviewed by Zinn-Justin (1989) indicate excellent agreement, not only between these two field-theoretic expansions but also with other statistical-mechanical formulations and with experimental data on a variety of condensed matter systems. For reasons which will become apparent, it is difficult, if not impossible, to construct the ϵ expansion consistently in thermal field theory, and we shall rely on a variant of the fixed-dimension approach.

Of particular interest in thermal field theory is the effective potential (thermodynamically, the free energy density). Anticipating a close analogy with thermal field theory, we now estimate this potential for the Ginzburg–Landau–Wilson model in three dimensions and verify that the transition is indeed second order. The point at issue is the following. At a given temperature (which here means a given value of μ^2) the thermodynamically stable state of our system corresponds to an expectation value v of the field which minimizes the effective potential $V_{\mathrm{eff}}(v)$. At low temperatures, the global minimum is at a non-zero value of v, say $\hat{v}$, corresponding to spontaneous symmetry breaking, while at sufficiently high temperature it is at $v = 0$. If $\hat{v}$ approaches 0 continuously at the transition temperature, the transition is of second order. Another possibility is that, for some range of temperature, the effective potential possesses two minima, at $v = \hat{v}$ and at $v = 0$. In that case, we would identify a first-order transition, where the expectation value changes discontinuously from $\hat{v}$ to zero, at a temperature where $V_{\mathrm{eff}}(\hat{v}) = V_{\mathrm{eff}}(0)$. We have identified the critical temperature as $\mu^2 = 0$, where the second derivative of V_{eff} vanishes. Suppose that at this temperature, the global minimum of V_{eff} is at some non-zero $\hat{v}$. As μ^2 increases, the point $v = 0$ becomes a local minimum, and $V_{\mathrm{eff}}(\hat{v})$ increases. This would indicate a first-order transition at some positive value of μ^2.

In three dimensions, a direct calculation at one-loop order yields

$$V_{\mathrm{eff}} = \frac{1}{2}\mu^2\left(1+\frac{\lambda}{8\pi}\right)v^2 + \frac{\lambda\kappa}{4!}\left(1+\frac{3\lambda}{16\pi}\right)v^4 - \frac{1}{12\pi}m^3(v) \tag{2.23}$$

where v denotes the exact expectation value of the renormalized field and

$$m^2(v) = \mu^2 + \tfrac{1}{2}\kappa^{\epsilon}\lambda v^2 \tag{2.24}$$

is the mass appearing in the unperturbed propagator for the shifted field $\phi - v$. At $\mu^2 = 0$, the last term in (2.23) is proportional to $-|v|^3$. Taken at face value, this indicates a global minimum at non-zero v, and hence a first-order transition. However, (2.23) cannot be taken at face value, because the effective expansion parameter is $(\kappa/m(v))\lambda$ and higher-order terms are increasingly infrared divergent at small m. To determine the true behaviour of $V_{\mathrm{eff}}(v)$ when $\mu^2 = 0$ and v is small, we must use the renormalization group to control these infrared divergences.

The effective potential satisfies a renormalization-group equation similar to (2.7) which, when $\mu^2 = 0$, has the solution

$$V_{\mathrm{eff}}(\lambda, 0, v, \kappa) = (\xi\kappa)^d V_{\mathrm{eff}}(\bar{\lambda}(\xi), 0, \bar{v}(\xi)(\xi\kappa)^{1-d/2}, 1) \tag{2.25}$$

with $\bar{v}(\xi) = P^{1/2}(\xi)v$. To isolate the infrared divergences in the prefactor, the appropriate choice of ξ is now given by the condition

$$\bar{v}(\xi)(\xi\kappa)^{1-d/2} = 1 \tag{2.26}$$

and we find

$$V_{\text{eff}}(\lambda, 0, v, \kappa) \sim |v|^{1+\delta} \tag{2.27}$$

where the exponent δ is

$$\delta = \frac{d+2-\eta}{d-2+\eta}. \tag{2.28}$$

At one-loop order, $\eta(\lambda)$ is zero, and in three dimensions, the best estimates give $\eta \equiv \eta(\lambda^*) \approx 0.04$ (Zinn-Justin 1989). If we ignore η, we find that $V_{\text{eff}} \sim v^4$ in four dimensions, but $V_{\text{eff}} \sim v^6$ in three dimensions. If we now use (2.23) to estimate the right-hand side of (2.25) explicitly, we find $\xi\kappa \approx v^2$ and

$$V_{\text{eff}} \approx \frac{\lambda^*}{4!}\left[1 - \left(\frac{\lambda^*}{2\pi^2}\right)^{1/2} + \frac{3}{16\pi}\lambda^*\right]v^6 \tag{2.29}$$

with $\lambda^* = 16\pi/3$. (Explicit details of the renormalization-group functions will be given in a generalised form below.) While the overall factor v^6 in this expression is certainly correct, the coefficient is not particularly reliable, for the reasons discussed above. (Note, however, that the rather large value of λ^* must be taken in conjunction with a factor $1/2\pi^2$ which arises from each loop integral, giving an effective expansion parameter $\lambda^*/2\pi^2 \approx 0.85$.) Nevertheless, we see that the effective potential of the massless theory has a single minimum at $v = 0$, so that the transition is indeed second order, as all theoretical treatments and experimental investigations agree.

3. Scalar field theory at finite temperature

In the imaginary time formalism, $\lambda\phi^4$ field theory at finite temperature $T = \beta^{-1}$ is described by the Euclidean action

$$S = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi/\partial\tau)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu_0^2\phi^2 + \frac{\lambda_0}{4!}\phi^4\right] \tag{3.1}$$

where d is again the number of spatial dimensions, and ϕ is periodic in τ with period β. (We have suppressed the subscript indicating a non-renormalized field to avoid a proliferation of indices below.) It is equivalent to a $(d+1)$-dimensional Ginzburg–Landau–Wilson model, for a system of finite extent β in the last dimension. Such models have been studied for a long time in the context of classical and quantum statistical mechanics (see, for example, Barber and Fisher (1973), Young (1975), Hertz (1976), Lawrie (1978a, b), Lawrie and Fisher (1978)) and it is found that their asymptotic critical behaviour is the same as that of the d-dimensional model. This is because the infrared singularities which dominate the critical region depend on cooperative effects on the length scale of the correlation length (or inverse mass) which becomes much larger than β.

For this reason, it will be helpful to examine the high-temperature limit $\beta \to 0$ of the action (3.1). Since the field is periodic, we may expand it as

$$\phi(x,\tau) = \sum_{n=-\infty}^{\infty} \exp(\mathrm{i}\nu_n\tau)\phi_n(x) \tag{3.2}$$

where the Matsubara frequencies are $\nu_n = 2\pi n/\beta$. When $\beta \to 0$, all field modes except $n = 0$ have large effective masses, $m_n^2 = \nu_n^2 + \mu_0^2$, and are suppressed. In this limit, we obtain an effective d-dimensional action

$$\tilde{S} = \int \mathrm{d}^d x \left[\frac{1}{2}(\nabla\tilde{\phi})^2 + \frac{1}{2}\mu_0^2\tilde{\phi}^2 + \frac{\tilde{\lambda}_0}{4!}\tilde{\phi}^4\right] \tag{3.3}$$

where

$$\tilde{\phi}(x) = \beta^{-1/2}\phi_0(x) \tag{3.4}$$

$$\tilde{\lambda}_0 = \beta^{-1}\lambda_0. \tag{3.5}$$

We shall be particularly interested in Green functions whose external frequencies are all zero, and for the unrenormalized one-particle irreducible functions, we obtain

$$\Gamma_0^{(N)}(0;\{p\};\lambda_0,\mu_0^2,\beta) \xrightarrow{\beta\to 0} \beta^{N/2-1}\tilde{\Gamma}_0^{(N)}(\{p\};\tilde{\lambda}_0,\mu_0^2) \tag{3.6}$$

where $\tilde{\Gamma}_0^{(N)}$ is the function calculated from the action (3.3).

The evaluation of Feynman diagrams is most often accomplished by using the Matsubara representation, with propagators $(\nu_n^2 + k^2 + \mu^2)^{-1}$. For the purpose of evaluating Green functions with external frequencies equal to zero, we find it more convenient to avoid frequency sums by using the imaginary time representation, with the propagator

$$\begin{aligned} g_k(\tau-\tau') = \frac{1}{2\omega_k}\{&[(n_k+1)\mathrm{e}^{-\omega_k(\tau-\tau')} + n_k\mathrm{e}^{\omega_k(\tau-\tau')}]\theta(\tau-\tau') \\ &+ [(n_k+1)\mathrm{e}^{\omega_k(\tau-\tau')} + n_k\mathrm{e}^{-\omega_k(\tau-\tau')}]\theta(\tau'-\tau)\} \end{aligned} \tag{3.7}$$

where $\omega_k = (k^2+\mu^2)^{1/2}$ and n_k denotes the Bose–Einstein occupation number

$$n_k = \left[\exp(\beta\omega_k) - 1\right]^{-1}. \tag{3.8}$$

Each vertex carries an imaginary time argument. Because of the periodicity, the imaginary time of one (arbitrary) vertex in a diagram may be set to zero, while all others are integrated between 0 and β. For example, the integral corresponding to the two-loop diagram of figure 1(b) is

$$\begin{aligned} &\int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{(2\pi)^{2d}} \int_0^\beta \mathrm{d}\tau\, g_{k_1}(\tau) g_{k_2}(\tau) g_{k_3}(\tau) \\ &\quad = \frac{1}{4}\int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{(2\pi)^{2d}} \frac{n_1 n_2 n_3}{\omega_1\omega_2\omega_3}\left[\frac{\mathrm{e}^{\beta(\omega_1+\omega_2+\omega_3)} - 1}{\omega_1+\omega_2+\omega_3} + \frac{3(\mathrm{e}^{\beta(\omega_1+\omega_2)} - \mathrm{e}^{\beta\omega_3})}{\omega_1+\omega_2-\omega_3}\right] \end{aligned} \tag{3.9}$$

where ω_1 denotes ω_{k_1}, etc and $k_3 = k_1 + k_2$. It is useful to express the integrand of this and similar expressions as a polynomial of order s (in this case $s = 2$) in the n_i:

$$\frac{1}{4}\int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{(2\pi)^{2d}} \frac{1}{\omega_1\omega_2\omega_3}\left[\frac{1}{\omega_1+\omega_2+\omega_3}(1+n_1+n_2+n_3+n_1n_2+n_2n_3+n_3n_1) + \frac{3}{\omega_1+\omega_2-\omega_3}(n_3+n_2n_3+n_3n_1-n_1n_2)\right]. \tag{3.10}$$

The leading term, with no n_i, reproduces the corresponding integral in the zero temperature, $(d+1)$-dimensional theory. In the limit $\beta \to 0$, we have $n_i \to (\beta\omega_i)^{-1}$. The terms of order s combine to give a factor of β^{-s} times the corresponding integral in the theory defined by (3.3). It is not hard to see that the factors β^{-s} are just such as to reproduce the powers of β in (3.5) and (3.6).

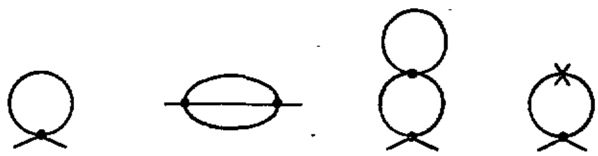

Figure 1. Low-order contributions to the propagator.

4. Calculation of the critical temperature

We asserted in the last section that the critical behaviour of the finite-temperature field theory (3.1) is the same as that of the d-dimensional Ginzburg–Landau–Wilson-like model (3.3) to which it reduces in the high-temperature limit. The cleanest way to analyse the infrared singularities in this d-dimensional model is, as we have seen, by means of an expansion about $d = 4$. However, the ultraviolet behaviour of the original model (3.1) is the same as that of the zero-temperature, $(d + 1)$-dimensional theory, which is not renormalizable for $d > 3$. Therefore, the ϵ expansion is not available to us, if we wish to remain within the class of renormalizable field theories. In particular, we would like to estimate the critical temperature of the theory in terms of the renormalized mass and coupling constant of the zero-temperature, $(d + 1)$-dimensional theory, which would be impracticable within an expansion about $d = 4$. In the condensed-matter context, one can indeed obtain some useful information about the dimensional crossover by working near $d = 4$ (Lawrie 1978b) and introducing a momentum cutoff to eliminate ultraviolet divergences. However, the usefulness of this approach is limited to analysing infrared behaviour inside the critical region. The information which would allow one to estimate the critical temperature of the underlying condensed-matter system is not contained in the effective Ginzburg–Landau–Wilson Hamiltonian. In what follows, therefore, we will set

$$d = 3 - \epsilon \tag{4.1}$$

with a view to dimensional regularization of ultraviolet divergences but our interest will be in the limit $\epsilon \to 0$, and the analysis of infrared behaviour will be via the three-dimensional approach.

Rather than try to estimate the critical temperature directly, we generalize the discussion of section 2, by finding the critical value, $\mu_{0c}^2(\lambda_0, \beta)$, of μ_0^2 for which $\Gamma_0^{(2)}(p = 0)$ vanishes at a given temperature β^{-1}. This relation can then be inverted to find the critical temperature corresponding to a given mass. In the first instance, we write

$$\mu_0^2 = \mu_{0c}^2(\lambda_0, \infty) + \Delta\mu_{0c}^2(\lambda_0, \beta) + r_0 \tag{4.2}$$

so that $\Gamma_0^{(2)}(p = 0)$ vanishes at $r_0 = 0$. A further multiplicative mass renormalization will then be needed, both to eliminate ultraviolet divergences and to exponentiate the infrared singularities. As before, our perturbation series uses r_0 as the squared mass in the unperturbed propagator, while the other two terms in (4.2) contribute counterterm vertices.

To see how the critical mass is to be identified, we consider in detail the evaluation of the Feynman diagram of figure 1(a), which is the one-loop contribution to the propagator. It is proportional to the integral $J = J_1 + J_2$, where

$$J_1 = \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\omega_k} = \int \frac{\mathrm{d}^{d+1} k}{(2\pi)^{d+1}}\frac{1}{(k^2 + r_0)} \tag{4.3}$$

$$J_2 = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\omega_k(\mathrm{e}^{\beta\omega_k} - 1)} \tag{4.4}$$

and $\omega_k = (k^2 + r_0)^{1/2}$. We are interested in the behaviour of these integrals at small values of r_0. The first may be written as

$$J_1 = \int \frac{\mathrm{d}^{d+1} k}{(2\pi)^{d+1}}\frac{1}{k^2} - r_0\int \frac{\mathrm{d}^{d+1} k}{(2\pi)^{d+1}}\frac{1}{k^2(k^2 + r_0)}. \tag{4.5}$$

The first term, which formally represents the value of J_1 at $r_0 = 0$, is an ill defined integral, which contributes to $\mu_{0c}^2(\lambda_0, \infty)$. The second is convergent, for $1 < d < 3$, and we can legitimately rescale k by a factor of $r_0^{1/2}$, to give

$$J_1 = \int \frac{\mathrm{d}^{d+1} k}{(2\pi)^{d+1}}\frac{1}{k^2} - r_0^{1-\epsilon/2}\int \frac{\mathrm{d}^{d+1} k}{(2\pi)^{d+1}}\frac{1}{k^2(k^2 + 1)}. \tag{4.6}$$

For d near 3, J_2 yields an infrared convergent integral at $r_0 = 0$. Subtracting this value, we get a remainder which vanishes at $r_0 = 0$, in which k can again be rescaled, giving

$$J_2 = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{|k|(\mathrm{e}^{\beta|k|} - 1)} + r_0^{1-\epsilon/2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\left[\frac{1}{\bar{\omega}_k(\mathrm{e}^{\beta\sqrt{r_0}\bar{\omega}_k} - 1)} - \frac{1}{|k|(\mathrm{e}^{\beta\sqrt{r_0}|k|} - 1)}\right] \tag{4.7}$$

where $\bar{\omega}_k = (k^2+1)^{1/2}$. Although the second integral does not have a power series expansion in $\beta\sqrt{r_0}$, its leading term as $\beta\sqrt{r_0} \to 0$ is a finite integral times $(\beta\sqrt{r_0})^{-1}$.

This kind of analysis can be repeated for every integral contributing to $\Gamma_0^{(2)}$. For a term of order λ_0^n, we will find an expression of the form

$$I = I_0 + r_0^{1-n\epsilon/2} I_1 + I_2(\beta) + r_0^{1-n\epsilon/2} I_3(\beta\sqrt{r_0}). \tag{4.8}$$

The integral I_0 is an ill defined quantity, which will contribute to $\mu_{0c}^2(\lambda_0, \infty)$, while I_1 has poles at $\epsilon = 0$, which will be absorbed by a multiplicative mass renormalization. The integral $I_2(\beta)$ is convergent for $d = 3 - \epsilon$, but generally has poles at $\epsilon = 0$, while $I_3(\beta\sqrt{r_0})$ diverges as $(\beta\sqrt{r_0})^{-n}$ as $\beta\sqrt{r_0} \to 0$, perhaps with an extra logarithmic factor. This last assertion follows from the fact that we must recover the propagator of the high-temperature theory (3.3) with coupling constant $\tilde{\lambda}_0 = \beta^{-1}\lambda_0$. In this theory, all integrals are convergent, with the exception of those shown in figure 1. The diagram of figure 1(b) is logarithmically divergent in three dimensions, and its I_3 diverges as $\ln(\beta\sqrt{r_0})/(\beta^2 r_0)$. Higher-order diagrams containing this subintegral will inherit the extra logarithm. To phrase this point more accurately, the limit which led from the $(d+1)$-dimensional action (3.1) to the d-dimensional action (3.3) is not strictly valid when the d-dimensional theory has divergences, and we recover the three-dimensional theory only up to logarithmic corrections.

The integrals of type I_0 in (4.8) are precisely the quadratically divergent integrals of the $(d+1)$-dimensional, zero temperature theory, and will be absorbed in $\mu_{0c}^2(\lambda_0, \infty)$. We now claim that the temperature-dependent contribution to the critical mass, $\Delta\mu_{0c}^2(\lambda_0, \beta)$ in (4.2) is obtained by subtracting the integrals of type $I_2(\beta)$. To be sure, the remaining integrals of type $I_3(\beta\sqrt{r_0})$ diverge as $r_0 \to 0$, and increasingly so at higher orders of perturbation theory. However, these divergences are of the same kind as those encountered in section 2, and we expect to be able to exponentiate them into an overall positive power of r_0 as in (2.3). In the next section, we shall exhibit a renormalization-group scheme which accomplishes this, and confirm that the critical mass is given by the above prescription, up to an isolated contribution from the diagram of figure 1(b), which may indicate an undetermined, non-perturbative correction. For now, we assume that this is so, and obtain the two-loop approximation to the critical temperature.

We need only the diagrams of figures 1(a) and 1(b). Those of figures 1(c) and 1(d) (where the cross denotes the one-loop value of μ_{0c}^2) cancel at $r_0 = 0$. Consider, then, the expression for the two-loop integral given in (3.10). Only the terms with one n give infrared-convergent integrals at $r_0 = 0$, and it is precisely these terms which contribute to $\Delta\mu_{0c}^2(\lambda_0, \beta)$. Taking into account also the one-loop contribution (4.4), we obtain

$$\mu_{0c}^2 = \mu_{0c}^2(\lambda_0, \infty) - \frac{\lambda_0}{2}\int \frac{d^d k}{(2\pi)^d}\frac{1}{|k|(e^{\beta|k|}-1)} \times \left[1 - \frac{\lambda_0}{2}\int \frac{d^d k'}{(2\pi)^d}\frac{|k'|+|k''|}{|k'||k''|[(|k'|+|k''|)^2 - k^2]}\right] \tag{4.9}$$

where $k'' = k + k'$. The subintegral in the two-loop part has a logarithmic ultraviolet divergence (at $d = 3$) and the expression clearly need renormalization.

To obtain a physically meaningful result, we need to express the critical temperature in terms of parameters of the renormalized zero-temperature theory. To this end, we first define a renormalized mass $\hat{m}$ as the pole of the zero-temperature propagator:

$$\Gamma_0^{(2)}(p^2 = -\hat{m}^2) = 0. \tag{4.10}$$

We also need a renormalized coupling constant, which is independent of any arbitrary renormalization scale, and the following is a convenient, if unconventional choice. We normalize the field $\phi(x)$ so that the renormalized propagator has unit residue at its pole:

$$\left.\frac{\partial}{\partial p^2}\Gamma^{(2)}(p)\right|_{p^2=-\hat{m}^2} = 1 \tag{4.11}$$

and denote by $\hat{v}$ its zero-temperature expectation value. Then our renormalized dimensionless coupling constant is

$$\hat{\lambda} = 3\hat{m}^{2-\epsilon}/\hat{v}^2. \tag{4.12}$$

We now have

$$\mu_0^2 = \mu_{0c}^2(\lambda_0, \infty) - \tfrac{1}{2}\hat{Z}_m(\hat{\lambda})\hat{m}^2 \tag{4.13}$$

where $\hat{Z}_m = 1 + \mathrm{O}(\hat{\lambda})$ is the renormalization constant implied by (4.10). Theoretically, at least, we can adjust μ_0^2 to have the value (4.9) corresponding to a critical temperature β^{-1}. The corresponding value of $\hat{m}^2$, say $\hat{m}_c^2(\hat{\lambda}, \beta)$ is found by equating the right-hand sides of (4.9) and (4.13). After evaluating $\hat{Z}_m$, expressing λ_0 in terms of $\hat{\lambda}$ and $\hat{m}$, and setting $d = 3$, we find the relation

$$\beta^2\hat{m}_c^2 = \frac{\hat{\lambda}}{12} - \frac{\hat{\lambda}^2}{96\pi^2}\ln(\beta\hat{m}_c) + c\hat{\lambda}^2 \tag{4.14}$$

where c is a finite integral which we are unable to compute in closed form. On inverting this relation, we estimate the critical temperature corresponding to a zero-temperature mass $\hat{m}$ as

$$T_c^2 = \hat{m}^2\left[\frac{\hat{\lambda}}{12} - \frac{1}{192\pi^2}\hat{\lambda}^2\ln\hat{\lambda} + \mathrm{O}(\hat{\lambda}^2)\right]^{-1}. \tag{4.15}$$

The term of order $\hat{\lambda}^2$ could be computed numerically, but its actual value is unenlightening, and depends on the precise definition of $\hat{\lambda}$. The leading term agrees with the old result of Dolan and Jackiw (1974), though this is commonly expressed in terms of a mass parameter μ^2, such that $\mu_0^2 = \mu^2 + \mathrm{O}(\lambda) = -\hat{m}^2/2 + \mathrm{O}(\lambda)$.

5. Renormalization and the effective potential

It remains to construct a renormalization-group scheme which will correctly exponentiate the infrared singularities in the critical region. Let us first consider what is required. We hope to obtain a generalization of (2.16), which relates a Green function with a very small mass to one with a non-vanishing mass. In the finite-temperature theory, we have an extra parameter, β with dimension $(\text{mass})^{-1}$, so the new relation must be of the form

$$\Gamma^{(2)}(0; \lambda, \mu^2, \beta, \kappa) = P(\xi)(\xi\kappa)^2\Gamma^{(2)}(0; \bar{\lambda}(\xi), 1, \xi\beta\kappa, 1). \tag{5.1}$$

The infrared singularity will be correctly isolated in the prefactor on the right-hand side, provided that $\bar{\lambda}(\xi)$ approaches an infrared-stable fixed point as $\xi \to 0$ and that the remaining Green function remains finite and non-zero in the limit $\xi\beta\kappa \to 0$.

Now, we anticipate that the critical singularity is that of the d-dimensional, high-temperature theory, with bare coupling constant $\tilde{\lambda}_0 = \beta^{-1}\lambda_0$ as in (3.5), so it should be the renormalized version of $\tilde{\lambda}_0$ which approaches λ^*. To achieve this, the normalization condition which replaces (2.22) should, at least for high temperatures, have the form

$$\tilde{\Gamma}^{(4)}(0; \tilde{\lambda}, \kappa^2, \kappa) = \kappa^{1+\epsilon}\tilde{\lambda} \tag{5.2}$$

where now $\epsilon = 3-d$. According to (3.6), this implies for the renormalized finite-temperature vertex function

$$\Gamma^{(4)}(0; \lambda, \kappa^2, \beta, \kappa) \xrightarrow{\beta\to 0} (\beta\kappa)\kappa^{\epsilon}\tilde{\lambda}. \tag{5.3}$$

At low temperatures, on the other hand, a normalization condition such as

$$\Gamma^{(4)}(0; \lambda, \kappa^2, \beta, \kappa) = \kappa^{\epsilon}\lambda \tag{5.4}$$

is appropriate for removing the ultraviolet singularities of the full $(d+1)$-dimensional theory. To interpolate between these two descriptions, we introduce a function $g(\beta\kappa)$ such that

$$g(\infty) = 1 \qquad g(\beta\kappa) \overset{\beta\to 0}{\approx} \beta\kappa \tag{5.5}$$

but which is otherwise arbitrary, and set

$$\lambda_0 = Z_\lambda(\lambda, \beta\kappa)\kappa^{\epsilon}\lambda g(\beta\kappa). \tag{5.6}$$

The function $Z_\lambda(\lambda, \beta\kappa)$ will be chosen in such a way that $\Gamma^{(4)}$ is finite at $\epsilon = 0$ and that

$$\lim_{\beta\kappa\to 0} (\beta\kappa)^{-1}\Gamma^{(4)}(0; \lambda, \kappa^2, \beta, \kappa) = \kappa^{\epsilon}\lambda \tag{5.7}$$

but any prescription which satisfies these conditions will serve. The result of calculating any physical quantity should, of course, be independent both of the value of the renormalization scale κ and of the particular function $g(\cdot)$. However, the truncation error incurred at a given order of perturbation theory is well known to be renormalization-scheme-dependent, and explicit results may well depend on these quantities. In the calculations we shall present, though, it will not prove necessary to make any specific choice of $g(\cdot)$.

Similarly, the additive mass renormalization in (4.2) will be supplemented by a multiplicative renormalization. However, there is a difficulty associated with the logarithmic divergence in figure 1(*b*) in the three-dimensional theory, which requires an extra additive renormalization. We therefore express μ_0^2 as

$$\mu_0^2 = \mu_{0c}^2(\lambda_0, \beta) + f(\lambda_0, \beta, r_0) + Z_\mu(\lambda, \beta\kappa)\mu^2 \tag{5.8}$$

with

$$r_0 = Z_\mu(\lambda, \beta\kappa)\mu^2 \tag{5.9}$$

where $f(\lambda_0, \beta, r_0)$ and Z_μ are chosen so that $\Gamma^{(2)}$ is finite at $\epsilon = 0$ and

$$\lim_{\beta\kappa\to 0} \Gamma^{(2)}(0; \lambda, \kappa^2, \beta, \kappa) = \kappa^2. \tag{5.10}$$

In general, the analogous modification of (2.21) will define a wavefunction renormalization, but our explicit calculations from now on will be restricted to one-loop order, where no wavefunction renormalization is required.

It should be clear that these conditions serve to make $\Gamma^{(2)}$ on the right of (5.1) finite in the limit $\xi \to 0$ as required. Moreover, the expansion parameter in the renormalized perturbation theory is $\lambda g(\beta\kappa)$. So, consider the form (4.8) of integrals contributing to $\Gamma^{(2)}$, and replace r_0 by κ^2, as is appropriate for determining the Z_i. The terms of type I_0 and I_2 are absorbed into μ_{0c}^2. The terms of type I_1 will contribute to Z_μ, and so will most of the remaining integrals of type I_3. Each of these type I_3 integrals will be multiplied by a factor $g^n(\beta\kappa)$ and the combination has a finite limit as $\beta\kappa \to 0$, except that the troublesome diagram in figure 1(*b*) would cause logarithmic divergences.

This diagram involves an integral of type I_3 whose leading behaviour as $\beta\sqrt{r_0} \to 0$ in the unrenormalized theory is of the form

$$r_0^{1-\epsilon} I_3(\beta\sqrt{r_0}) \sim \beta^{2\epsilon-2}\left[\frac{c_1}{\epsilon}\left((\beta^2 r_0)^{-\epsilon} - 1\right) + c_2\right] \tag{5.11}$$

where c_1 and c_2 are finite constants. In the limit $\beta \to 0$ with $\epsilon > 0$, this becomes $\beta^{-2} r_0^{-\epsilon} c_1/\epsilon$. This ultraviolet pole occurs, of course, in the three-dimensional Ginzburg–Landau–Wilson model, and is the only divergence in the dimensionally regularized theory. It has been discussed in detail by Bagnuls and Bervillier (1983, 1985; see also Symanzik (1973)), who conclude that it must be absorbed into μ_{0c}^2 (equation (2.5)) rather than into Z_μ. Indeed, the role of Z_μ is to renormalize insertions of the composite operator ϕ^2. This could alternatively be achieved by replacing (5.9) with a condition on $\partial\Gamma^{(2)}/\partial\mu^2$, which contains no pole at two-loop order, though it naturally contains the subdiagram in question at higher orders. If we take the limit $\epsilon \to 0$ in (5.11), on the other hand, we get the expression

$$\beta^{-2}(-c_1 \ln(\beta^2 r_0) + c_2). \tag{5.12}$$

It will shortly become clear that we do not want residual logarithmic divergences in Z_μ as $\beta\kappa \to 0$. We therefore choose f in (5.8) to remove the logarithm:

$$f(\lambda_0, \beta, r_0) = c'_1 \lambda_0^2 \beta^{-2} \ln(\beta^2 r_0) \tag{5.13}$$

where c'_1 includes the appropriate weight factor. Because this subtraction is made at the level of the bare theory, it will subtract the logarithm from every occurrence of the subdiagram in the renormalized theory.

With this prescription, all of the renormalization factors Z_i contain the poles of the $(4-\epsilon)$-dimensional theory, but otherwise have finite limits as $\beta\kappa \to 0$. Let us see whether the required fixed point exists. At one-loop order, a suitable pair of renormalization factors is

$$Z_\lambda = 1 + \frac{3}{2}\lambda g(\beta\kappa)\int \frac{d^d k}{(2\pi)^d}\left[\frac{1}{4(k^2+1)^{3/2}} + (\beta\kappa)^{-1}\frac{1}{(k^2+1)^2}\right] \tag{5.14}$$

$$Z_\mu = 1 + \lambda g(\beta\kappa)\int \frac{d^d k}{(2\pi)^d}\left[\frac{1}{(8-4\epsilon)(k^2+1)^{3/2}} + (\beta\kappa)^{-1}\frac{1}{(1-\epsilon)(k^2+1)^2}\right]. \tag{5.15}$$

The renormalization-group functions are finite at $\epsilon = 0$ and we find

$$W(\lambda, \beta\kappa) \equiv \kappa \frac{d\lambda}{d\kappa}\bigg|_{\lambda_0,\mu_0,\beta} = -\lambda\beta\kappa\frac{d\ln g}{d(\beta\kappa)} + \frac{3}{16\pi^2}\left(1 + \frac{\pi}{\beta\kappa}\right)\lambda^2 g \tag{5.16}$$

$$\sigma(\lambda, \beta\kappa) \equiv \kappa\frac{d\ln\mu^2}{d\kappa}\bigg|_{\lambda_0,\mu_0,\beta} = \frac{1}{8\pi\beta\kappa}\lambda g. \tag{5.17}$$

The running coupling constant in (5.1) is the solution of

$$\xi \frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}\xi} = W(\bar{\lambda}, \xi\beta\kappa) \tag{5.18}$$

with $\bar{\lambda}(1) = \lambda$. It is given by

$$\bar{\lambda}(\xi) = \frac{\lambda g(\beta\kappa)}{g(\xi\beta\kappa)} \left[1 + \frac{3\lambda g(\beta\kappa)}{16\pi^2}\left(\frac{\pi}{\xi\beta\kappa} - \frac{\pi}{\beta\kappa} - \ln\xi\right)\right]^{-1} \tag{5.19}$$

and we see that, as $\xi \to 0$, it approaches the same fixed-point value $\lambda^* = 16\pi/3$ that we found for the three-dimensional Ginzburg–Landau–Wilson model. The running mass is defined as in (2.13), except that σ now has an additional dependence on $\xi'\beta\kappa$. It will be convenient to have an estimate in closed form for $\bar{\mu}^2$. To this end, we omit the $\ln\xi$ in (5.19), which is small compared with $\pi/\xi\beta\kappa$ for $0 \leqslant \xi \leqslant 1$, which is the range of interest. With this approximation, we obtain

$$\bar{\mu}^2(\xi) \approx \mu^2 \left[\frac{\alpha(\beta\kappa)^{-1} + 1 - \alpha}{\alpha(\xi\beta\kappa)^{-1} + 1 - \alpha}\right]^{\sigma} \tag{5.20}$$

with $\sigma = 2/3$ and $\alpha = 3\lambda g(\beta\kappa)/16\pi$. When ξ is small, therefore, we have $\bar{\mu}^2(\xi) \sim \mu^2\xi^\sigma$ where, in general, $\sigma = \sigma(\lambda^*, 0)$. Note that this fixed-point exponent would not exist if we were to include in Z_μ the logarithmically divergent terms discussed above.

We now return to our earlier discussion of the critical temperature. To determine the renormalization factors Z_i, we considered the theory with $\mu^2 = \kappa^2$, for which (5.8) becomes

$$\mu_0^2 = \mu_{0c}^2(\lambda_0, \beta) + f(\lambda_0, \beta, r_0(\kappa)) + Z_\mu(\lambda, \beta\kappa)\kappa^2 \tag{5.21}$$

where $r_0(\kappa) = Z_\mu(\lambda, \beta\kappa)\kappa^2$. From the normalization condition (5.10), it is clear that $\Gamma^{(2)}$ vanishes when $\kappa^2 = 0$. To be confident that we have identified the critical temperature correctly, however, we must assure ourselves that the last two terms in (5.21) vanish when $\kappa^2 = 0$. As we have seen, the coefficient of λ^n in Z_μ has a finite limit (treating the poles of the zero-temperature theory as unimportant constants for this purpose). Moreover, at a fixed value of λ_0, λ approaches λ^* as $\kappa \to 0$. Term by term, therefore, $Z_\mu\kappa^2$ does vanish. This point can be carried a little further by using the renormalization group to estimate the behaviour of Z_μ. We have

$$\left(\kappa\frac{\partial}{\partial\kappa} + W(\lambda, \beta\kappa)\frac{\partial}{\partial\lambda}\right)\ln Z_\mu(\lambda, \beta\kappa) = -\sigma(\lambda, \beta\kappa) \tag{5.22}$$

which integrates to give

$$Z_\mu(\bar{\lambda}(\xi), \xi\beta\kappa) = \exp\left[-\int_1^\xi \frac{\mathrm{d}\xi'}{\xi'}\sigma(\bar{\lambda}(\xi'), \xi'\beta\kappa)\right] Z_\mu(\lambda, \beta\kappa). \tag{5.23}$$

We conclude that, when $\kappa \to 0$ with λ_0 fixed, $Z_\mu \sim \kappa^{-\sigma}$ and

$$Z_\mu(\lambda, \beta\kappa)\kappa^2 \sim \kappa^{2-\sigma}. \tag{5.24}$$

Our one-loop estimate above gave $\sigma = 2/3$, so it would appear that $(2 - \sigma)$ is positive, and $Z_\mu\kappa^2$ does indeed vanish as $\kappa \to 0$. In fact, $(2 - \sigma)$ is equal to ν^{-1}, where ν is the correlation-length exponent, for whose value in three dimensions the best available estimates give $\nu \approx 0.63$. The remaining term $f(\lambda_0, \beta, r_0(\kappa))$ is more problematic, however. If we insert $\lambda_0 = \lambda g(\beta\kappa) Z_\lambda(\lambda, \beta\kappa)$ and $r_0 = Z_\mu(\lambda, \beta\kappa)\kappa^2$ into (5.13) and expand in powers of the renormalized coupling constant, the overall factor $g^2(\beta\kappa)$ ensures that f does vanish term by term. On the other hand, the original expression diverges when $r_0 \to 0$ with λ_0 and β fixed. We conclude that, although the critical temperature we calculated in the last section is correct within renormalized perturbation theory, there may be a non-perturbative correction, which we are unable to determine.

We are now in a position to study the effective potential in the critical region. Takahashi (1985) and Carrington (1992) claim that the form of the effective potential is such as to indicate a first-order transition, but we shall show that this claim is erroneous, for precisely the reason explained in section 2. Observe, first of all that, because of the counterterm $\mu^2_{0c}(\lambda_0, \beta)$ in (5.8), the mass parameter μ^2 in our renormalized perturbation theory vanishes at the critical temperature. At one-loop order, Carrington achieves the same effect by resumming ring diagrams, but at higher orders, all other self-energy diagrams also contribute to $\mu^2_{0c}(\lambda_0, \beta)$. The effective potential may be calculated by the same method as before. At one-loop order, with $d = 3$, and with the approximation that the effective mass $(\mu^2 + \lambda g v^2/2)^{1/2}$ is much smaller than the temperature $T = \beta^{-1}$, we find

$$\begin{aligned} V_{\text{eff}}(\lambda, \mu^2, \beta, \kappa, v) = {} & \frac{1}{2}\mu^2\left[1 + \lambda g\left(\frac{1}{8\pi}(\beta\kappa)^{-1} - \frac{1}{16\pi^2}\ln\beta\kappa\right)\right]v^2 \\ & + \frac{\lambda g}{4!}\left[1 + \lambda g\left(\frac{3}{16\pi}(\beta\kappa)^{-1} - \frac{3}{16\pi^2}\ln\beta\kappa - \frac{3}{32\pi^2}\right)\right]v^4 \\ & - \frac{1}{12\pi}\beta^{-1}\left(\mu^2 + \frac{1}{2}\lambda g v^2\right)^{3/2}. \end{aligned} \tag{5.25}$$

With some minor differences arising from our renormalization scheme, this is the same as the result given by Carrington. The term of interest is the last one which, when $\mu^2 = 0$, is proportional to $-|v|^3$. As explained in section 2, however, we cannot conclude that V_{eff} has a local maximum at $v = 0$ and a minimum at some non-zero value of v, because the infrared singularities which occur at higher orders make the expression (5.25) completely unreliable. As before, the renormalization group can be used to exponentiate these singularities. We obtain the relation

$$V_{\text{eff}}(\lambda, 0, \beta, \kappa, v) = (\xi\kappa)^{d+1} V_{\text{eff}}(\bar\lambda(\xi), 0, \xi\beta\kappa, 1, [g(\xi\beta\kappa)]^{-1/2}) \tag{5.26}$$

where ξ must now be determined by the condition

$$(\xi\kappa)^{d-1} = g(\xi\beta\kappa)\bar v^2(\xi) \tag{5.27}$$

to ensure that the effective mass remains non-zero. At one-loop order, there is no wavefunction renormalization, so $\bar v(\xi) = v$. Using (5.25) to evaluate the right-hand side of (5.26) and setting $d = 3$, we obtain for small v

$$V_{\text{eff}} \approx \beta^{-1}\frac{\lambda^*}{4!}\left[1 - \left(\frac{\lambda^*}{2\pi^2}\right)^{1/2} + \frac{3}{16\pi}\lambda^*\right](\beta^{1/2}v)^6. \tag{5.28}$$

As might have been expected, this is identical to (2.29), except that v has been replaced by $\beta^{1/2}v$, which reflects the relation (3.5) between the fields of the d- and $(d+1)$-dimensional theories, and an overall factor of β^{-1}. The latter factor may be accounted for by considering that the effective potential is the effective action, evaluated with a constant classical field, divided by the volume. In the case of the three-dimensional Ginzburg–Landau–Wilson model, this volume is the three-volume $\int d^3x$ whereas in thermal field theory it is the four-volume $\beta \int d^3x$.

6. The order parameter: an illustrative calculation

We illustrate how our formalism may be applied to calculate quantities of physical interest by obtaining an expression for the order parameter (that is, the expectation value $\langle\phi\rangle$) as a function of temperature. This expectation value is not directly measurable, but it is of interest here, insofar as it approaches zero continuously as $T \to T_c$, illustrating the second-order character of the transition.

In dealing with phenomena occurring below the critical temperature, where symmetry is spontaneously broken, it is convenient to redefine the mass parameter μ^2 by a factor $-1/2$, in which case (5.8) becomes

$$\mu_0^2 = \mu_{0c}^2(\lambda_0, \beta) - \tfrac{1}{2} Z_\mu(\lambda, \beta\kappa)\mu^2. \tag{6.1}$$

Then the expectation value $v = \langle\phi\rangle$ has the form

$$v^2 = \frac{3\mu^2}{\kappa^\epsilon \lambda g(\beta\kappa)} V(\lambda, \mu^2, \beta, \kappa) \tag{6.2}$$

where $V = 1 + \mathrm{O}(\text{one-loop})$. The renormalization group, together with dimensional analysis implies a relation analogous to (2.16), namely

$$v^2(\lambda, \mu^2, \beta, \kappa) = (\xi\kappa)^{2-\epsilon} v^2(\bar\lambda(\xi), \bar\mu^2(\xi)/(\xi\kappa)^2, \xi\beta\kappa, 1) \tag{6.3}$$

provided that we ignore η, which is zero at one-loop order. In the now-familiar way, we regularize the infrared divergences in the loop corrections by choosing ξ to satisfy the condition $\bar\mu^2(\xi) = (\xi\kappa)^2$, whereupon (6.2) becomes

$$v^2 = \frac{3\bar\mu^{2-\epsilon}(\xi)}{\bar\lambda(\xi) g(\xi\beta\kappa)} V(\bar\lambda(\xi), 1, \xi\beta\kappa, 1). \tag{6.4}$$

Approximate solutions for $\bar\lambda(\xi)$ and $\bar\mu^2(\xi)$ were given in (5.19) and (5.20). For brevity, we write these as $\bar\mu^2(\xi) = \mu^2 M(\xi)$ and $\bar\lambda(\xi) g(\xi\beta\kappa) = \lambda g / L(\xi)$, where g means $g(\beta\kappa)$. We now have

$$v^2 = \frac{3\mu^{2-\epsilon}}{\lambda g} M^{1-\epsilon/2}(\xi) L(\xi) V(\bar\lambda(\xi), 1, \xi\beta\kappa, 1). \tag{6.5}$$

We would like to convert this expression into a form which depends only on temperature, and on the mass and coupling constant of the zero-temperature theory. To this end, we reintroduce the zero-temperature parameters $\hat m$ and $\hat\lambda$ defined in (4.10)–(4.13). Moreover, v is the expectation value of a field whose normalization depends, in general, on the arbitrary renormalization scale κ. To get a more meaningful result, we therefore define $\hat v_\beta$ to be the expectation value at temperature β^{-1} of the field whose normalization is fixed at zero temperature by (4.11). Thus, $\hat v_\beta \to \hat v$ as $T \to 0$. We now have two renormalization schemes, which are related by the parameters of the bare theory. That is,

$$\lambda_0 = \kappa^\epsilon \lambda g Z_\lambda = \hat m^\epsilon \hat\lambda \hat Z_\lambda \tag{6.6}$$

$$\mu_0^2 - \mu_{0c}^2(\lambda_0, \infty) = \Delta\mu_{0c}^2(\lambda_0, \beta) - \tfrac{1}{2}\mu^2 Z_\mu = -\tfrac{1}{2}\hat m^2 \hat Z_m. \tag{6.7}$$

The latter relation can be written at one-loop order as

$$\mu^2 = \left(1 - \frac{T^2}{T_c^2}\right) \hat{m}^2 \frac{\hat{Z}_m}{Z_\mu} \tag{6.8}$$

but (4.14) shows that the factor $(1 - T^2/T_c^2)$ will be modified at higher orders by logarithms of T/T_c.

The resulting expression for $\hat{v}_\beta$ contains ratios of Z factors, which must have finite limits at $\epsilon = 0$. We can therefore take this limit to obtain

$$\hat{v}_\beta^2 = \hat{v}^2 \left(1 - \frac{T^2}{T_c^2}\right) M(\xi) L(\xi) U(\xi) \tag{6.9}$$

where

$$U(\xi) = \frac{Z_\phi Z_\lambda \hat{Z}_m}{\hat{Z}_\phi \hat{Z}_\lambda Z_\mu} V(\bar{\lambda}(\xi), 1, \xi\beta\kappa, 1). \tag{6.10}$$

If we were able to evaluate $\hat{v}_\beta$ exactly, it would necessarily be independent both of κ and of the function $g(\cdot)$. Indeed, the whole renormalization-group analysis would be redundant. As it is, however, we have to find M, L and U independently, each at some finite order of perturbation theory. This means that the truncation error we incur depends on κ and $g(\cdot)$ and so, therefore, does our result. The dependence on $g(\cdot)$ can be eliminated by observing that in (6.9) λg occurs only in one-loop terms. According to (6.6), therefore, λg can be replaced by $\hat{\lambda}$, the error being formally of a higher order than that to which we are working, and this is what we do. There does not appear to be any simple strategy for removing the dependence on κ, so we must choose a value for it. Fortunately, the requirement that $\hat{v}_\beta = \hat{v}$ at $T = 0$ (or $\beta = \infty$) is sufficient to fix this value. Unsurprisingly, we find that the appropriate value is

$$\kappa = \hat{m}. \tag{6.11}$$

Our final result for $\hat{v}_\beta$ is a little cumbersome to write down. It may be expressed as

$$\hat{v}_\beta^2 = \hat{v}^2 \xi^2 L(\xi) U(\xi). \tag{6.12}$$

For $U(\xi)$ we find

$$U(\xi) = 1 + \frac{3}{32\pi^2}(\bar{\lambda}(\xi) g(\xi\beta\hat{m}) - \hat{\lambda}) - \frac{1}{16\pi\beta\hat{m}}(\xi^{-1}\bar{\lambda}(\xi) g(\xi\beta\hat{m}) - \hat{\lambda}) \\ - \bar{\lambda}(\xi) g(\xi\beta\hat{m}) I(\xi\beta\hat{m}) \tag{6.13}$$

where

$$I(y) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[\frac{1}{\bar{\omega}_k (\mathrm{e}^{y\bar{\omega}_k} - 1)} - \frac{1}{|k|(\mathrm{e}^{y|k|} - 1)} \right] \tag{6.14}$$

with $\bar{\omega}_k = (k^2 + 1)^{1/2}$. The quantity $\bar{\lambda}(\xi) g(\xi\beta\hat{m})$ is now given by $\hat{\lambda}/L(\xi)$, where

$$L(\xi) = 1 + \frac{3\hat{\lambda}}{16\pi^2}\left(\frac{\pi}{\xi\beta\hat{m}} - \frac{\pi}{\beta\hat{m}} - \ln\xi\right) \tag{6.15}$$

and ξ is determined by the condition

$$\xi^2 = \left(1 - \frac{T^2}{T_c^2}\right)\left[\frac{\alpha(\beta\hat{m})^{-1} + 1 - \alpha}{\alpha(\xi\beta\hat{m})^{-1} + 1 - \alpha}\right]^\sigma \tag{6.16}$$

where now $\alpha = 3\hat{\lambda}/16\pi$ and σ is still equal to 2/3.

It would now be possible to compute $\hat{v}_\beta$ using numerical methods to evaluate the integral (6.14) and to solve (6.16) for ξ. When T_c is expressed in the form (4.15), one finds that the ratio $\hat{v}_\beta/\hat{v}$ is a function of the two dimensionless variables $\hat{\lambda}$ and $\beta\hat{m}$. The detailed numerical form of this function is not particularly enlightening, however, and we content ourselves with describing the limits $T \to 0$ and $T \to T_c$, which can be found analytically. In the limit $T \to 0$ (and $\beta \to \infty$), the solution of (6.16) is $\xi = 1$ and $I(\beta\hat{m})$ vanishes. One then easily finds that $\hat{v}_\beta = \hat{v}$ as advertised. To study the limit $T \to T_c$, it is convenient to define

$$t = \left(1 - \frac{T^2}{T_c^2}\right). \tag{6.17}$$

For small t, we find from (6.16) that $\xi \sim t^{1/(2-\sigma)}$. We then find that $L(\xi) \sim \xi^{-1}$ and $U(\xi)$ approaches a finite constant. Overall, therefore, we obtain

$$\hat{v}_\beta^2 \sim \xi \sim t^{2\beta} \tag{6.18}$$

where the exponent β is given by $\beta = 1/2(2-\sigma) = 3/8$. We recall that $1/(2-\sigma)$ is the correlation-length exponent ν. In general, if one takes into account the non-trivial wavefunction renormalization which arises at higher orders, giving rise to the Fisher exponent η, the value of β in d spatial dimensions is

$$\beta = \tfrac{1}{2}\nu(d - 2 + \eta) \tag{6.19}$$

(see, for example, Brézin *et al* (1976), Zinn-Justin (1989)), and the best estimate of its value in three dimensions is $\beta \approx 0.33$.

7. Gauge theories

A detailed study of phase transitions in gauge theories at finite temperature is beyond the scope of this paper, but several observations can be made. The simplest gauge theory is scalar quantum electrodynamics, whose three-dimensional version constitutes the Ginzburg–Landau–Wilson model of superconductivity. Experimentally, the superconducting phase transition always appears to be second order, as the tree-level potential would indicate. Nevertheless, it was suggested by Halperin *et al* (1974) that the transition might in fact be first order, although the magnitude of the order-parameter discontinuity would be very small in a real superconductor.

Their argument was closely related to a well known study of scalar QED by Coleman and Weinberg (1973), which establishes the possibility of spontaneous symmetry breaking by radiative corrections. In the one-loop effective potential of the Ginzburg–Landau–Wilson model or, equivalently, of the zero-temperature field theory, a single gauge loop contributes a term proportional to $-|v|^3$ in three dimensions or to $v^4 \ln v$ in four dimensions. As in (2.33), this is not in itself a reliable indication of a first-order transition. However, the renormalization-group argument we have employed in this paper also breaks down, because the required infrared-stable fixed point does not exist. More generally, one can consider a model in which $N/2$ complex scalar fields are coupled to the same Abelian gauge field. Within the ϵ expansion about four dimensions, one finds that a fixed point exists only for $N > 365.9 + O(\epsilon)$. The absence of the expected fixed point has often been taken as a signal of a first-order transition. Indeed, one can still use the renormalization group to control infrared singularities, and hence construct an improved approximation to the effective potential (Lawrie 1982) and this appears to confirm that the transition is of first order.

In the absence of unforseen circumstances, one would expect the phase transition in a $(d+1)$-dimensional thermal field theory to be of the same kind as that in the corresponding d-dimensional Ginzburg–Landau–Wilson model, as we have seen in detail for the scalar theory. On this basis, it seems likely that the phase transition in the standard model, say, might be of first order, as has been claimed by several authors, though the method of analysis used, for example, by Carrington (1992) is not adequate to establish this, and a detailed renormalization-group study is needed for any particular model.

Unfortunately, the matter does not rest here. A theory with continuous symmetry, such as the $O(N)$-symmetric scalar theory can be cast in the form of a nonlinear σ-model (formally, by taking the limit $\lambda \to \infty$ and $\mu^2 \to -\infty$ in such a way that the weight factor $\exp[-\mu^2\phi^2/2 - \lambda\phi^4/4!]$ becomes $\delta(\phi^2 - v^2)$). This theory is expected to have the same phase transition as the original one (see, for example, Zinn-Justin (1989)). In the nonlinear theory, however, the appropriate expansion parameter is $\epsilon = d - 2$ (Brézin and Zinn-Justin 1976a, b). Renormalization-group techniques can again be applied, though in a somewhat different form, and the critical temperature appears as an infrared-unstable fixed point. For the Abelian Higgs model, this approach was studied by Lawrie and Athorne (1983), who found that the required fixed point exists for any N, indicating a second-order transition. Thus, it appears that scalar electrodynamics has a first-order transition for spatial dimensionalities near four, but a second-order transition near two dimensions. What happens in the physically relevant case of three spatial dimensions has not, to our knowledge, been settled, and we know of no way of obtaining a conclusive answer. Quite probably, the same unsatisfactory situation obtains in connection with other gauge theories.

8. Summary and discussion

At a second-order phase transition, the effective mass of a field theory vanishes or, equivalently, the characteristic distance over which correlations decay becomes infinite. A $(3 + 1)$-dimensional field theory in thermal equilibrium is, in effect, a Euclidean system of finite extent in one of its dimensions, and the infrared singularities which characterize its critical behaviour are those of the three-dimensional theory. In the Ginzburg–Landau–Wilson model, which is essentially obtained by ignoring the finite dimension, renormalization-group methods allow the infrared divergences of standard perturbation theory to be resummed, by relating Green functions of the critical theory to those of the non-critical theory, for which perturbation theory is more reliable. In its most systematic form, this resummation relies on an expansion about four infinite dimensions, where the infrared singularities are logarithmic. On the other hand, once one knows that this is possible, direct calculations in three dimensions can be undertaken, though they are generally less accurate at low orders than the ϵ expansion. In finite-temperature field theory, an expansion about $(4+1)$ dimensions is not practicable (at least, in any obvious way) since ultraviolet divergences make the theory non-renormalizable, but we have shown how the fixed-dimension approach can be generalized to deal with this case.

We have shown how the critical temperature of the scalar theory can be calculated, at any order of perturbation theory, in terms of parameters of the zero-temperature theory. The renormalization group does not enter directly into this calculation, which involves only infrared-finite contributions to the two-point function, but it does assure us that the infrared divergences will not disturb the result. We found, however, that there may be a non-perturbative correction to our result, which we are unable to determine. The renormalization group is of vital importance in obtaining a reliable estimate of the effective potential. A direct calculation of $V_{\rm eff}(v)$ in perturbation theory seems to show that, at the temperature T_c for which its second derivative vanishes at $v = 0$, there is a global minimum at some non-zero v. This would indicate a first-order transition at a temperature somewhat higher than T_c. On using the renormalization group to resum infrared singularities, however, one finds that this appearance is misleading, and that the true minimum is indeed at $v = 0$. Thus, the transition is actually of second order, and occurs at T_c. This conclusion is confirmed by an explicit calculation of the expectation value of the field as a function of temperature, which shows that this expectation value vanishes continuously, with the expected universal power law, as $T \to T_c$.

We believe that these conclusions are definitive for scalar field theory, although explicit calculations at higher orders than we have considered might provide a valuable check. Certainly, a determination of the non-perturbative contribution to the critical temperature is desirable. Possibly, some modification of the renormalization scheme used here might allow it to be found self-consistently, but we have not succeeded in doing this. In gauge theories the situation is less certain. The Ginzburg–Landau superconductor lacks, at least near four spatial dimensions, the infrared-stable fixed point which is normally associated with a second-order transition. The indications are that the transition in this model is of first order near four spatial dimensions, but of second order near two dimensions. Almost certainly, the transition in $(3+1)$-dimensional scalar electrodynamics is identical to that in the three-dimensional superconductor, whose order cannot be reliably determined, and the same is probably true of other gauge theories.

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